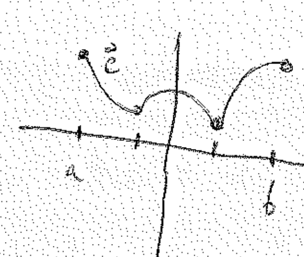
Ch. 5 Integration along paths

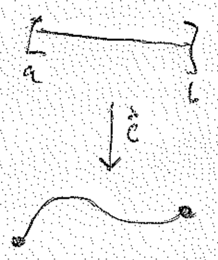
Sec 5.1 Paths and parametrizations

- **Def** A path $\vec{c}: [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is $\underline{C^1}$ if its component functions are C^1, i.e., have continuous partial derivatives.

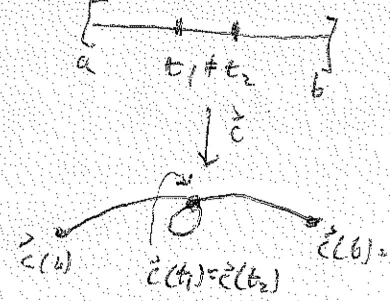
- **Def** A path is piecewise C^1 if $[a,b]$ can be broken into subintervals so that $\vec{c}$ is C^1 on each subinterval.



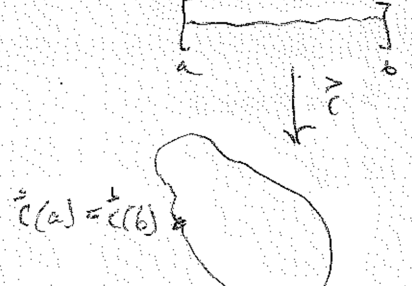
- **Def**



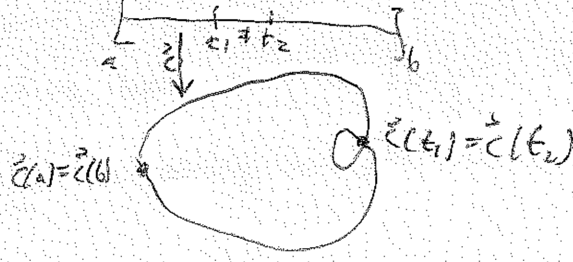
simple curve
(doesn't intersect itself)



a curve that is not simple
(intersects itself $\to$ not 1-1)



simple closed curve
(1-1 on $[a,b)$)



a closed curve that is not a simple closed curve
(not 1-1 on $[a,b)$)

Sec 5.2 Path integrals of real-valued functions

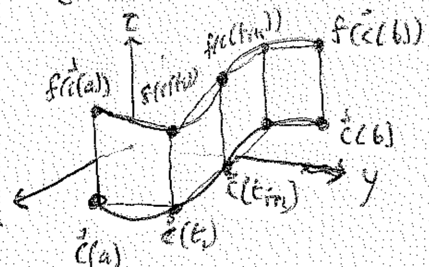
• **Def** Let $\vec{c}:[a,b]\to\mathbb{R}^2$ be a C^1 path, and let $f:\mathbb{R}^2\to\mathbb{R}$ be a function so that the composition $f(c(t))$ is continuous on $[a,b]$.

The path integral of f along $\vec{c}$ is

$$\int_{\vec{c}} f\,ds = \int_a^b f(c(t))\,\|c'(t)\|\,dt = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$$

• **Interpretation** If f is positive,

$\int_{\vec{c}} f\,ds$ = area of region along $\vec{c}$ and below f:



Why? Area $\approx \sum_{i=1}^{n} f(\vec{c}(t_i))\,\|\vec{c}(t_{i+1})-\vec{c}(t_i)\|$

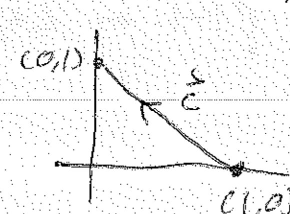
$$= \sum_{i=1}^{n} f(\vec{c}(t_i))\left\|\frac{\vec{c}(t_{i+1})-\vec{c}(t_i)}{t_{i+1}-t_i}\right\|\,\|t_{i+1}-t_i\|$$

$$\longrightarrow \int_a^b f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt = \int_{\vec{c}} f\,ds$$

as $n\to\infty$.

Notation For a path integral along a closed path, sometimes $\oint_{\vec{c}} f\,ds$ is written in place of $\int_{\vec{c}} f\,ds$.

Example Let $f(x,y)=2x+y$. Compute $\int_{\vec{c}} f\,ds$, with $\vec{c}$ as shown.



Soln First parametrize the curve:

$c(t) = (1,0) + t(0-1,1-0),\ t\in[0,1]$

$c(t) = (1-t,t),\ t\in[0,1]$

Then $c'(t) = (-1,1)$,

so $\|c'(t)\| = \sqrt{(-1)^2+1^2} = \sqrt{2}$.

Thus $\int_{\vec{c}} f\, ds = \int_0^1 f(\vec{c}(t))\, \|\vec{c}\,'(t)\|\, dt = \sqrt{2}\int_0^1 f(1-t, t)\, dt$

$$= \sqrt{2}\int_0^1 2(1-t) + t\, dt$$

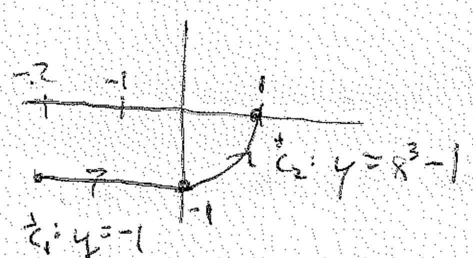
$$= \sqrt{2}\int_0^1 2 - t\, dt$$

$$= \sqrt{2}\left[2t - \frac{t^2}{2}\right]_0^1 = \boxed{\frac{3\sqrt{2}}{2}}.$$

Theorem Let $\vec{c}$ and f be as in the definition of $\int_{\vec{c}} f\, ds$. Let $\vec{c}^*$ be another parametrization of the same curve as $\vec{c}$. Then $\int_{\vec{c}} f\, ds = \int_{\vec{c}^*} f\, ds$.

Example Let $\vec{c}(t) = (1,0) + t(-1,1)$, $t \in [0,1]$.
Let $\vec{c}^*(t) = (1,0) + 2t(-1,1)$, $t \in [0, \frac{1}{2}]$.
One can check that $\int_{\vec{c}} f\, ds = \int_{\vec{c}^*} f\, ds = \frac{3\sqrt{2}}{2}$.

Example Evaluate $\int_{\vec{c}} 4x^3\, ds$ where $\vec{c}$ is as shown.



Soln Parametrize $\vec{c}_1$ and $\vec{c}_2$:
$c_1(t) = (t, -1)$, $t \in [-2, 0]$
$c_2(t) = (t, t^3 - 1)$, $t \in [0, 1]$

$$\int_{\vec{c}} f\, ds = \int_{\vec{c}_1} f\, ds + \int_{\vec{c}_2} f\, ds.$$

$$\int_{\vec{c}_1} f\, ds = \int_{-2}^0 4t^3\sqrt{1^2 + 0^2}\, dt = \left[t^4\right]_{-2}^0 = -16$$

$$\int_{\vec{c}_2} f\, ds = \int_0^1 4t^3\sqrt{1^2 + (3t^2)^2}\, dt = \int_0^1 4t^3\sqrt{1 + 9t^4}\, dt = \frac{2}{27}\left(10^{3/2} - 1\right)$$

(u-sub: $u = 1 + 9t^4$, $du = 36t^3\, dt$)

Thus $\int_{\vec{c}} f\, ds = -16 + \frac{2}{27}\left(10^{3/2} - 1\right)$.

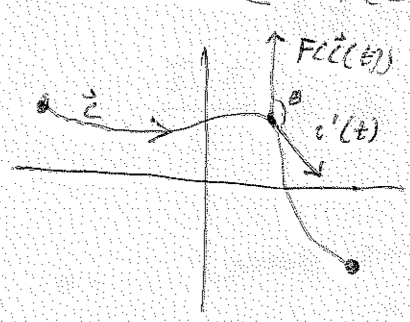
Sec 5.3 Path integrals of vector fields

- **Def** Let $\vec{c}:[a,b]\to\mathbb{R}^2$ (or $\mathbb{R}^3$) be a C^1 path, and let $F:\mathbb{R}^2\to\mathbb{R}$ (or $\mathbb{R}^3\to\mathbb{R}^3$) be a vector field such that $F(\vec{c}(t))$ is continuous on $[a,b]$.

  The path integral of F along $\vec{c}$ is

$$\int_{\vec{c}} F\cdot d\vec{s} = \int_a^b F(\vec{c}(t))\cdot c'(t)\,dt = \int_a^b \big(F_1(x(t),y(t)), F_2(x(t),y(t))\big)\cdot\big(x'(t),y'(t)\big)\,dt$$

- A path integral of a vector field is also called a line integral.
- If F is a force field, then $\int_{\vec{c}} F\cdot d\vec{s}$ = work done by $\vec{F}$ moving a particle along $\vec{c}$.
- Geometric intuition: $F(c(t))\cdot c'(t) = \|F(c(t))\|\,\|c'(t)\|\cos\theta$



$\leftarrow \theta > \frac{\pi}{2}$, so this dot product is negative.

Example Find the work done by the force $F(x,y,z)=(y+z,x,x)$ on a particle moving along the line segment from $(0,0,1)$ to $(1,0,1)$.

Soln First parametrize the curve: $\vec{c}(t)=(0,0,1)+t(1-0,0-0,1-1)$, $t\in[0,1]$

$$c(t)=(t,0,1),\ t\in[0,1].$$

so $c'(t)=(1,0,0)$.

Then $\int_{\vec{c}} F\cdot d\vec{c} = \int_0^1 F(\vec{c}(t))\cdot c'(t)\,dt = \int_0^1 F(t,0,1)\cdot(1,0,0)\,dt$

$$= \int_0^1 (0+1,t,t)\cdot(1,0,0)\,dt$$

$$= \int_0^1 1\,dt = \boxed{1}$$

• **Notation** We often use the notation $\boxed{\int_{\vec{c}} F \cdot d\vec{s} = \int_{\vec{c}} F_1\,dx + F_2\,dy + F_3\,dz}$

Why? $\int_{\vec{c}} F \cdot d\vec{s} = \int_a^b F \cdot c'\,dt = \int_a^b (F_1, F_2, F_3) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$

$$= \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

$$= \int_{\vec{c}} F_1\,dx + F_2\,dy + F_3\,dz$$

Example Compute $\int_{\vec{c}} x^2\,dx + y\,dy + 2yz$ along

$\vec{c}(t) = (1, t, -t^2)$, $t \in [0,1]$.

Soln $dx = 0$, $dy = dt$, $dz = -2t\,dt$

$$\int_{\vec{c}} x^2\,dx + y\,dy + 2yz\,dz = \int_0^1 [0 + y + 2yz(-2t)]\,dt$$

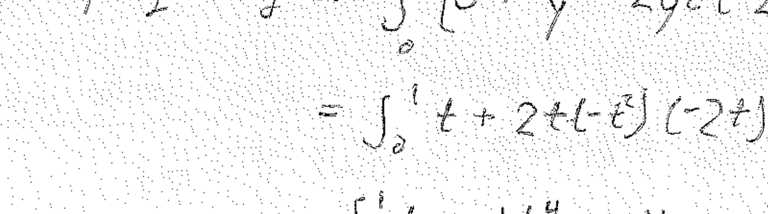
$$= \int_0^1 t + 2t(-t^2)(-2t)\,dt$$

$$= \int_0^1 t + 4t^4\,dt = \left[\frac{t^2}{2} + \frac{4t^5}{5}\right]_0^1 = \boxed{\frac{13}{10}}.$$

Theorem Let F and $\vec{c}$ be as in the definition of $\int_{\vec{c}} F \cdot d\vec{s}$

Let $\vec{c}^*$ be another parametrization of the same curve as $\vec{c}$.

Then $\int_{\vec{c}} F \cdot d\vec{s} = \begin{cases} \int_{\vec{c}^*} F \cdot d\vec{s} & \text{if } \vec{c} \text{ and } \vec{c}^* \text{ have the same orientation.} \\ -\int_{\vec{c}^*} F \cdot d\vec{s} & \text{if } \vec{c} \text{ and } \vec{c}^* \text{ have opposite orientations.} \end{cases}$



Note This is in contrast with path integrals of real-valued functions.

Example $F = (xy, z, x)$ $\quad \vec{c}_1(t) = (t, t, t^2)$ $t \in [0,1]$

$\vec{c}_2(t) = (1-t, 1-t, (1-t)^2)$, $t \in [0,1]$

Then $\int_{\vec{c}_1} F \cdot d\vec{s} = \int_0^1 F(t,t,t^2) \cdot (1,1,2t)\, dt$

$$= \int_0^1 (t^2, t^2, t) \cdot (1,1,2t)\, dt = \int_0^1 4t^2\, dt = \frac{4}{3}$$

and $\int_{\vec{c}_2} F \cdot d\vec{c} = \int_0^1 ((1-t)^2, (1-t)^2, 1-t) \cdot (-1, -1, -2(1-t))\, dt$

$$= \int_0^1 -4(1-t)^2\, dt = \frac{-4}{3}.$$

Sec 5.4 Path Integrals Independent of Path

- Def $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ (or $\mathbb{R}^3 \to \mathbb{R}^3$) is called a gradient vector field if there is a differentiable function $f : U \to \mathbb{R}$ so that $F = \nabla f$.

Example Show that $F = (y+z, x, x)$ is a gradient vector field.

Soln Want f so that $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y+z, x, x)$.

$\frac{\partial f}{\partial x} = y + z \longrightarrow f = xy + xz + g(y,z)$

$\longrightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} \qquad = x$

$\longrightarrow \frac{\partial g}{\partial y} = 0$

$\longrightarrow g(y,z) = h(z)$

$\longrightarrow f = xy + xz + h(z)$

$\longrightarrow \frac{\partial f}{\partial z} = x + h'(z) = x$

$\longrightarrow h'(z) = 0 \longrightarrow h(z) = C$, a constant

Thus F is a gradient vector field for $f = xy + xz + C$.

<u>Theorem</u> (Fundamental Theorem of Calculus for path integrals)
Let $f: \mathbb{R}^2$ (or $\mathbb{R}^3$) $\to \mathbb{R}$ be C^1 and let $\vec{c}: [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) be a piecewise C^1 path. Then

$$\int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a)).$$

<u>Proof</u> By the chain rule,

$$D(f \circ \vec{c})(t) = Df(\vec{c}(t)) \cdot D\vec{c}(t) = \nabla f(\vec{c}(t)) \cdot c'(t).$$

Thus
$$\begin{aligned} \int_{\vec{c}} \nabla f \cdot d\vec{s} &= \int_a^b \nabla f(\vec{c}(t)) \cdot \vec{c}'(t)\, dt \\ &= \int_a^b D(f \circ \vec{c})(t)\, dt \qquad \text{(by chain rule)} \\ &= f(\vec{c}(b)) - f(\vec{c}(a)) \qquad \text{(by the FTC).} \end{aligned}$$

<u>Example</u> Find $\int_{\vec{c}} (y+z)\,dx + x\,dy + x\,dz$, where $\vec{c}$ is the line segment from $(1,0,0)$ to $(1,0,1)$.

<u>Soln</u> We saw previously, for $F = (y+z, x, x)$, $f = xy + xz + C$, that $F = \nabla f$.

Thus
$$\begin{aligned} \int_{\vec{c}} F \cdot d\vec{s} = \int_{\vec{c}} \nabla f \cdot d\vec{s} &= f(\vec{c}(\vec{b})) - f(\vec{c}(\vec{a})) \\ &= \Big[xy + xz + C\Big]_{(1,0,0)}^{(1,0,1)} = \boxed{1}. \end{aligned}$$

• <u>Properties of Gradient Vector Fields</u>:

Let F be a C^1 vector field on an open, simply connected set $U \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$. The following are equivalent:

① F is a gradient vector field

② For any oriented, simple closed curve $\vec{c}$, $\int_{\vec{c}} F \cdot d\vec{s} = 0$

③ F is <u>path independent</u>: for any two oriented, simple curves $\vec{c}_1$ and $\vec{c}_2$ having the same endpoints, $\int_{\vec{c}_1} F \cdot d\vec{s} = \int_{\vec{c}_2} F \cdot d\vec{s}$.

④ curl $F = \vec{0}$ (in $\mathbb{R}^3$), or <u>scalar curl</u> $F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$.

- In physics, a force field is <u>conservative</u> where $F = -\nabla V$ where V is a potential function. We restate the properties in physics terms:

- <u>Theorem</u> With F as above, the following are equivalent:

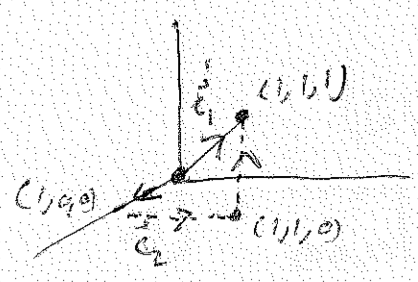
① F is conservative

② The work done by F along any oriented simple closed curve is 0.

③ The work done is path independent.

④ F is <u>irrotational</u>: (scalar) curl $F = 0$.

<u>Example</u> Compute $\int_{\vec{c}_1} F \cdot d\vec{s}$ and $\int_{\vec{c}_2} F \cdot d\vec{s}$, where $F = (2xy+z, x^2, x)$ along $\vec{c}_1$ and $\vec{c}_2$ as shown:



<u>Soln</u> Because $F = \nabla(x^2y + xz)$, the theorem says that $\int_{\vec{c}_1} F \cdot d\vec{s} = \int_{\vec{c}_2} F \cdot d\vec{s}$. We check.

$\vec{c}_1(t) = (t,t,t)$, $t \in [0,1]$

$\vec{c}_2(t) = (t,0,0) \cup (1,t,0) \cup (1,1,t)$, $t \in [0,1]$.

$$\int_{\vec{c}_1} F \cdot d\vec{s} = \int_0^1 (2 \cdot t \cdot t + t, t^2, t) \cdot (1,1,1)\, dt$$

$$= \int_0^1 3t^2 + 2t \, dt$$

$$= 2.$$

$$\int_{\vec{c}_2} F \cdot d\vec{s} = \int_{(t,0,0)} F \cdot d\vec{s} + \int_{(1,t,0)} F \cdot d\vec{s} + \int_{(1,1,t)} F \cdot d\vec{s}$$

$$= \int_0^1 (0,t^2,t)\cdot(1,0,0)\,dt + \int_0^1 (2t,1,1)\cdot(0,1,0)\,dt + \int_0^1 (2+t,1,1)\cdot(0,0,1)\,dt$$

$$= \int_0^1 0\,dt + \int_0^1 1\,dt + \int_0^1 1\,dt = 2.$$

Alternately, $\int_{\vec{c}} F \cdot d\vec{s} = \int_{\vec{c}} \nabla(x^2y + xz)\cdot d\vec{s}$

$$= \left[x^2y + xz\right]_{(0,0,0)}^{(1,1,1)} = 2.$$